
UNIT 10 HYPOTHESIS TESTING

Structure

- 10.0 Introduction
- 10.1 Objectives
- 10.2 Hypothesis
- 10.3 Key Concepts for Hypothesis Testing
 - 10.3.1 Critical Region
 - 10.3.2 Type-I and Type-II Errors
 - 10.3.3 Level of Significance
 - 10.3.4 One-Tailed and Two-Tailed Tests
 - 10.3.5 Concept of p-Value
- 10.4 General Procedure of Testing a Hypothesis
- 10.5 Testing of Hypothesis for Population Mean using Z-Test
- 10.6 Testing of Hypothesis for Difference of Two Population Means using Z-Test
- 10.7 Testing of Hypothesis for Population Proportions using Z-Test
- 10.8 Testing of Hypothesis for Population Mean using t-Test
- 10.9 Summary
- 10.10 Solutions /Answers
- 10.11 References

10.0 INTRODUCTION

In the previous units of this course, we have discussed one part of statistical inference, that is, estimation and we have learnt how we estimate the unknown population parameter(s) by using **point estimation** and **interval estimation**. In this block, we will focus on the second part of statistical inference which is known as **testing of hypothesis**.

In our day-to-day life, we see different commercials and advertisements on television, newspapers, magazines, etc. such as

- (i) The refrigerator of a certain brand saves up to 20% electric bill,
- (ii) The motorcycle of a certain brand gives 60 km/liter mileage,
- (iii) A detergent of a certain brand produces the cleanest wash,
- (iv) Ninety-nine out of hundred dentists recommend brand A toothpaste for their patients to save the teeth against cavity, etc.

Now, the question may arise in our mind “can such types of claims be verified statistically?” Fortunately, in many cases, the answer is “yes”.

The technique of testing such type of claims or statements or assumptions is known as testing of hypothesis. The truth or falsity of a claim or statement is never known unless we examine the entire population. But practically it is not possible in most situations so we take a random sample from the population under study and use the information contained in this sample to take the decision whether a claim is true or false.

This Unit covers in detail the concepts of hypothesis testing. It discusses the hypothesis testing of the population mean and population proportions using the Z-test. Further, it discusses the testing of hypotheses using the t-test. This unit also covers a topic on testing of hypothesis for two population means.

10.1 OBJECTIVES

After reading this unit, you should be able to:

- define a hypothesis;
- formulate the null and alternative hypotheses;
- explain what we mean by type-I and type-II errors;
- explore the concept of critical region and level of significance;
- define one-tailed and two-tailed tests;
- concept of p-value;
- applying the Z-test for testing the hypothesis about the population mean and difference of two population means;
- applying the Z-test for testing the hypothesis about the population proportions; and
- describe testing of hypothesis for the population mean using t-test.

10.2 HYPOTHESIS

In our day-to-day life, we see different commercials and advertisements in television, newspapers, magazines, etc. and if someone may be interested to test such type of claims or statement then we come across the problem of testing of hypothesis. For example,

- (i) a customer of motorcycle wants to test whether the claim about the motorcycle of the certain brand gives the average mileage 60 km/liter is true or false,
- (ii) the businessman of banana wants to test whether the average weight of a banana of Kerala is more than 200 gm,
- (iii) a doctor wants to test whether a new medicine is more effective for controlling high blood pressure than old medicine,
- (iv) an economist wants to test whether the variability in incomes differ in two populations,
- (v) a psychologist wants to test whether the proportion of literates between two groups of people is the same, etc.

In all the cases discussed above, the decision-maker is interested in making inference about the population parameter(s). However, he/she is not interested in estimating the value of the parameter(s) but he/she is interested in testing a claim or statement or assumption about the value of population parameter(s). Such a claim or statement is postulated in terms of hypothesis.

In statistics, a hypothesis is a statement or a claim or an assumption about the value of a population parameter (e.g., mean, median, variance, proportion, etc.).

Similarly, in the case of two or more populations, a hypothesis is a comparative statement or a claim or an assumption about the values of population parameters. (e.g., means of two populations are equal, variance of one population is greater than other, etc.). The plural of a hypothesis is hypotheses.

In hypothesis testing problems, first of all, we should be identifying the claim or statement or assumption or hypothesis to be tested and write it in the words. Once

the claim has been identified then we write it in symbolical form if possible. As in the above examples,

- (i) Customer of the motorcycle may write the claim or postulate the hypothesis “the motorcycle of the certain brand gives the average mileage 60 km/liter.” Here, the concern is the **average** mileage of the motorcycle so let μ represents the average mileage then our hypothesis becomes $\mu = 60$ km/ liter.
- (ii) Similarly, the businessman of banana may write the statement or postulate the hypothesis “the average weight of a banana of Kerala is greater than 200 gm.” So our hypothesis becomes $\mu > 200$ gm.
- (iii) The doctor may write the claim or postulate the hypothesis “ the new medicine is more effective for controlling high blood pressure than old medicine.” Here, we are concerned with the **average** effect of the medicines so let μ_1 and μ_2 represent the average effect of new and old medicines respectively on controlling blood pressure, then our hypothesis becomes $\mu_1 > \mu_2$.
- (iv) The economist may write the statement or postulate the hypothesis “ the variability in incomes differ in two populations.” Here, our concern is the **variability** in income so let σ_1^2 and σ_2^2 represent the variability in incomes in two populations respectively then our hypothesis becomes $\sigma_1^2 \neq \sigma_2^2$.
- (v) The psychologist may write the statement or postulate the hypothesis “the proportion of literates between two groups of people is same.” Here, we are concerning the **proportion** of literates so let P_1 and P_2 represent the proportions of literates of two groups of people respectively then our hypothesis becomes $P_1 = P_2$ or $P_1 - P_2 = 0$.

The hypothesis is classified according to its nature and usage, as we will discuss next.

Null and Alternative Hypotheses

As we have discussed in the last page that in hypothesis testing problems, first of all, we identify the claim or statement to be tested and write it in symbolical form. After that, we write the complement or opposite of the claim or statement in symbolical form. In our example of motorcycle, the claim is $\mu = 60$ km/liter then its complement is $\mu \neq 60$ km/liter. In (ii) the claim is $\mu > 200$ gm then its complement is $\mu \leq 200$ gm. If the claim is $\mu < 200$ gm then its complement is $\mu \geq 200$ gm. The claim and its complement are formed in such a way that they cover all possibilities of the value of the population parameter.

Once the claim and its complement have been established then we decide of these two which is the null hypothesis and which is the alternative hypothesis. The thumb rule is that the statement containing equality is the null hypothesis. That is, a hypothesis which contains symbols $=$ or $\leq$ or $\geq$ is taken as a null hypothesis and a hypothesis which does not contain equality i.e. contains $\neq$ or $<$ or $>$ is taken as an alternative hypothesis. The null hypothesis is denoted by H_0 and the alternative hypothesis is denoted by H_1 or H_A .

In our example of motorcycle, the claim is $\mu = 60$ km/liter and its complement is $\mu \neq 60$ km/liter. Since claim $\mu = 60$ km/liter contains equality sign so we take it as a null hypothesis and complement $\mu \neq 60$ km/liter as an alternative hypothesis, that is,

$$H_0: \mu = 60 \text{ km/liter and } H_1: \mu \neq 60 \text{ km/liter}$$

In our second example of banana, the claim is $\mu > 200$ gm and its complement is $\mu \leq 200$ gm. Since complement $\mu \leq 200$ gm contains equality sign so we take complement as a null hypothesis and claim $\mu > 200$ gm as an alternative hypothesis, that is,

$$H_0: \mu \leq 200 \text{ gm and } H_1: \mu > 200 \text{ gm}$$

Formally, these hypotheses are defined as

The hypothesis which we wish to test is called the null hypothesis.

According to Prof. R.A. Fisher,

“A null hypothesis is a hypothesis which is tested for possible rejection under the assumption that it is true.”

The hypothesis which complements to the null hypothesis is called the alternative hypothesis.

Note 1: Some authors use equality sign (=) in the null hypothesis instead of $\geq$ and $\leq$ signs.

The alternative hypothesis has two types:

- (i) Two-sided (tailed) alternative hypothesis
- (ii) One-sided (tailed) alternative hypothesis

If an alternative hypothesis gives the alternate of the null hypothesis in both directions (less than and greater than) of the value of the parameter specified in null hypothesis then it is known as two-sided alternative hypothesis and if it gives an alternate only in one direction (less than or greater than) only then it is known as one-sided alternative hypothesis. For example, if our alternative hypothesis is $H_1: \theta \neq 60$ then it is a two-sided alternative hypothesis because it means that the value of parameter θ is greater than or less than 60. Similarly, if $H_1: \theta > 60$ then it is a right-sided alternative hypothesis because it means that the value of parameter θ is greater than 60 and if $H_1: \theta < 60$ then it is a left-sided alternative hypothesis because it means that the value of parameter θ is less than 60.

In the testing procedure, we assume that the null hypothesis is true until there is sufficient evidence to prove that it is false. Generally, a hypothesis is tested with the help of a sample so the evidence about the claim comes from a sample. If there is enough sample evidence to suggest that the null hypothesis is false then we reject the null hypothesis and support the alternative hypothesis. If the sample fails to provide us sufficient evidence against the null hypothesis we are not saying that the null hypothesis is true because here, we take the decision on the basis of a random sample which is a small part of the population. To say that a null hypothesis is true, we must study all observations of the population under study. For example, if someone wants to test that the person of India has two online accounts then to prove that this is true we must check all the persons of India whereas to prove that it is false we require a person he/ she has one online account or no online account. So we can only say that there is not enough evidence against the null hypothesis.

Note 2: When we assume that a null hypothesis is true then we are actually assuming that the population parameter is equal to the value in the claim. In our example of motorcycle, we assume that $\mu = 60$ km/liter whether the null hypothesis is $\mu = 60$ km/liter or $\mu \leq 60$ km/liter or $\mu \geq 60$ km/liter.

10.3 KEY CONCEPTS FOR HYPOTHESIS TESTING

After describing the hypothesis, let us discuss some basic concepts needed for hypothesis testing.

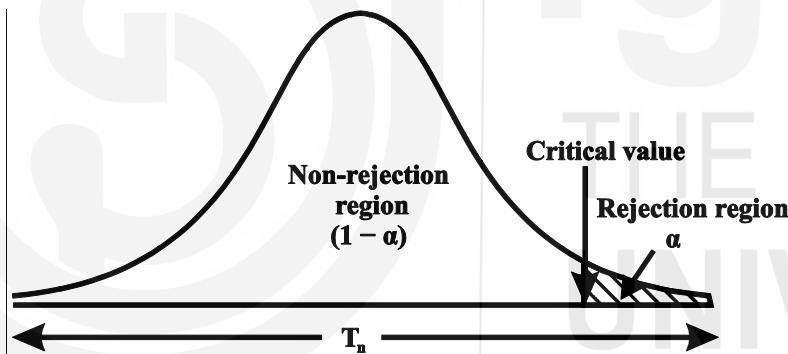
10.3.1 Critical Region

In order to test a hypothesis, the entire sample space is partitioned into two disjoint sub-spaces, say ω and $S - \omega = \bar{\omega}$. If calculated value of the test statistic lies in ω , then we reject the null hypothesis and if it lies in $\bar{\omega}$, then we do not reject the null hypothesis. The region ω is called a “**rejection region or critical region**” and the region $\bar{\omega}$ is called a “**non-rejection region**”. Therefore, we can say that

“A region in the sample space in which if the calculated value of the test statistic lies, we reject the null hypothesis then it is called critical region or rejection region.”

The rejection (critical) region lies in one-tail or two-tails on the probability curve of the sampling distribution of the test statistic its depends upon the alternative hypothesis. Therefore, three cases arise:

Case I: If the alternative hypothesis is right-sided such as $H_1: \theta > \theta_0$ or $H_1: \theta_1 > \theta_2$ then the entire critical or rejection region of size α lies on the right tail of the probability curve of the sampling distribution of the test statistic as shown in Fig. 10.1.



Critical value is a value or values that separate the region of rejection from the non-rejection region.

Fig. 10.1: Rejection Region for Right-sided Hypothesis

Case II: If the alternative hypothesis is left-sided such as $H_1: \theta < \theta_0$ or $H_1: \theta_1 < \theta_2$ then the entire critical or rejection region of size α lies on the left tail of the probability curve of the sampling distribution of the test statistic as shown in Fig. 10.2.

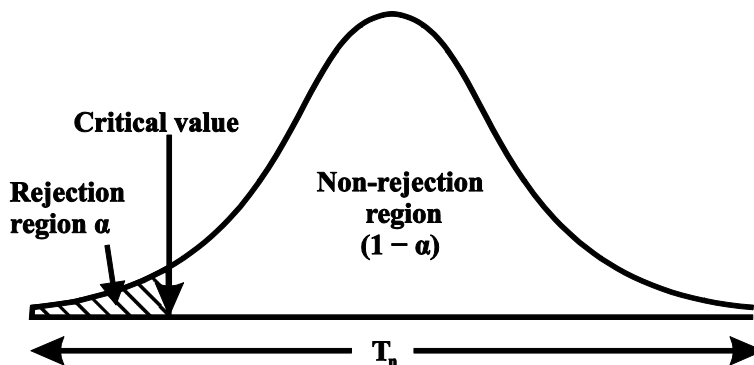


Fig. 10.2: Rejection Region for Left-sided Hypothesis

Case III: If the alternative hypothesis is two-sided such as $H_1: \theta \neq \theta_0$ or $H_1: \theta_1 \neq \theta_2$ then critical or rejection regions of size $\alpha/2$ lies on both tails

of the probability curve of the sampling distribution of the test statistic as shown in Fig. 10.3.

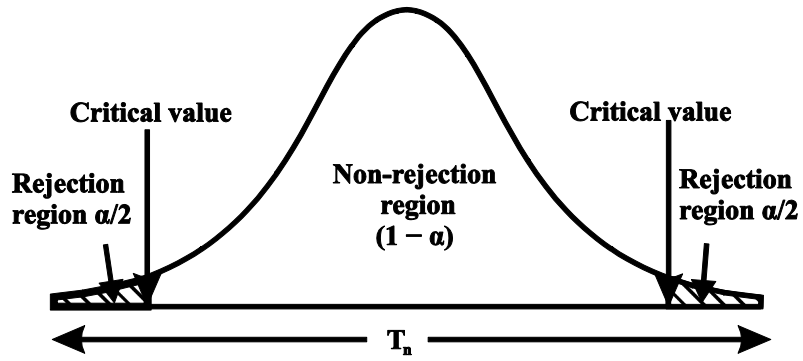


Fig. 10.3: Rejection Region for Two-sided Hypothesis

10.3.2 Type-I and Type-II Errors

In the last sub-section, we have discussed a rule that if the value of test statistic falls in rejection (critical) region then we reject the null hypothesis and if it falls in the non-rejection region then we do not reject the null hypothesis. A test statistic is calculated on the basis of observed sample observations. But a sample is a small part of the population about which decision is to be taken. A random sample may or may not be a good representative of the population.

A faulty sample misleads the inference (or conclusion) relating to the null hypothesis. For example, an engineer infers that a packet of screws is sub-standard when actually it is not. It is an error caused due to poor or inappropriate (faulty) sample. Similarly, a packet of screws may be inferred to be good when actually it is sub-standard. So we can commit two kinds of errors in testing a hypothesis which are summarised in Table 10.1 which is given below:

Table 10.1: Type of Errors

Decision	H_0 True	H_1 True
Reject H_0	Type-I Error	Correct Decision
Do not reject H_0	Correct Decision	Type-II Error

Let us take a situation where a patient suffering from high fever reaches to a doctor. And suppose the doctor formulates the null and alternative hypotheses as

H_0 : The patient is a malaria patient

H_1 : The patient is not a malaria patient

Then the following cases arise:

Case I: Suppose that the hypothesis H_0 is really true, that is, the patient actually a malaria patient and after observation, pathological and clinical examination, the doctor rejects H_0 , that is, he/ she declares him/ her a non-malaria-patient. It is not a correct decision and he/ she commits an error in the decision known as type-I error.

Case II: Suppose that the hypothesis H_0 is actually false, that is, the patient actually a non-malaria patient and after observation, the doctor rejects H_0 , that is, he/ she declares him/ her a non-malaria-patient. It is a correct decision.

Case III: Suppose that the hypothesis H_0 is really true, that is, the patient actually a malaria patient and after observation, the doctor does not reject H_0 , that is, he/ she declares him/ her a malaria-patient. It is a correct decision.

Case IV: Suppose that the hypothesis H_0 is actually false, that is, the patient actually a non-malaria patient and after observation, the doctor does not reject H_0 , that is, he/she declares him/her a malaria-patient. It is not a correct decision and he/she commits an error in the decision known as type-II error.

10.3.3 Level of Significance

So far in this unit, we have discussed the hypothesis, types of hypothesis, critical region and types of errors. In this section, we shall discuss a very useful concept “**level of significance**”, which plays an important role in decision making while testing a hypothesis.

The probability of type-I error is known as level of significance of a test. It is also called the **size of the test** or **size of the critical region**, denoted by α . Generally, it is pre-fixed as 5% or 1% level ($\alpha = 0.05$ or 0.01). As we have discussed in Section 10.3 that if calculated value of the test statistic lies in rejection (critical) region, then we reject the null hypothesis and if it lies in non-rejection region, then we do not reject the null hypothesis. Also, we note that when H_0 is rejected then automatically the alternative hypothesis H_1 is accepted. Now, one point of our discussion is how to decide critical value(s) or cut-off value(s) for a known test statistic.

If distribution of test statistic could be expressed into some well-known distributions like Z , χ^2 , t , F etc. then our problem will be solved and using the probability distribution of test statistic, we can find the cut-off value(s) that provides us critical area equal to 5% (or 1%).

Another viewpoint about the level of significance relates to the trueness of the conclusion. If H_0 does not reject at a level, say, $\alpha = 0.05$ (5% level) then a person will be confident that “concluding statement about H_0 ” is true with 95% assurance. But even then it may false with 5% chance. There is no cent-per cent assurance about the trueness of a statement made for H_0 .

As an example, if among 100 scientists, each draws a random sample and use the same test statistic to test the same hypothesis H_0 conducting the same experiment, then 95 of them will reach to the same conclusion about H_0 . But still, 5 of them may differ (i.e. against the earlier conclusion).

Similar argument can be made for, say, $\alpha = 0.01$ (=1%). It is like when H_0 is rejected at $\alpha = 0.01$ by a scientist, then out of 100 similar researchers who work together at the same time for the same problem, but with different random samples, 99 of them would reach to the same conclusion, however, one may differ.

10.3.4 One-Tailed and Two-Tailed Test

We have seen that rejection (critical) region lies at one-tail or two-tails on the probability curve of sampling distribution of the test statistic, depending on the form of alternative hypothesis. Similarly, the test of testing the null hypothesis also depends on the alternative hypothesis.

A test of testing the null hypothesis is said to be a two-tailed test if the alternative hypothesis is two-tailed whereas if the alternative hypothesis is one-tailed then a test of testing the null hypothesis is said to be a one-tailed test.

For example, if our null and alternative hypotheses are

$$H_0 : \theta = \theta_0 \text{ and } H_1 : \theta \neq \theta_0$$

then the test for testing the null hypothesis is two-tailed because the alternative hypothesis is two-tailed that means, the parameter θ can take value greater than θ_0 or less than θ_0 .

If the null and alternative hypotheses are

$$H_0 : \theta \leq \theta_0 \text{ and } H_1 : \theta > \theta_0$$

then the test for testing the null hypothesis is right-tailed because the alternative hypothesis is right-tailed.

Similarly, if the null and alternative hypotheses are

$$H_0 : \theta \geq \theta_0 \text{ and } H_1 : \theta < \theta_0$$

then the test for testing the null hypothesis is left-tailed because the alternative hypothesis is left-tailed.

The above discussion can be summarised in Table 10.2.

Table 10.2: Null and Alternative Hypotheses and Corresponding One-tailed and Two-tailed Tests

Null Hypothesis	Alternative Hypothesis	Types of Critical Region / Test
$H_0 : \theta = \theta_0$	$H_1 : \theta \neq \theta_0$	Two-tailed test having critical regions under both tails.
$H_0 : \theta \leq \theta_0$	$H_1 : \theta > \theta_0$	Right-tailed test having critical region under right tail only.
$H_0 : \theta \geq \theta_0$	$H_1 : \theta < \theta_0$	Left- tailed test having critical region under left tail only.

Let us do one example based on the type of tests.

Example 1: A company has replaced its original technology of producing electric bulbs by CFL technology. The company manager wants to compare the average life of bulbs manufactured by original technology and new technology CFL. Write appropriate null and alternate hypotheses. Also, say about one-tailed and two-tailed tests.

Solution: Suppose the average lives of original and CFL technology bulbs are denoted by μ_1 and μ_2 respectively.

If the company manager is interested just to know whether any significant difference exists in the average lifetime of two types of bulbs then null and alternative hypotheses will be:

$$H_0: \mu_1 = \mu_2 \quad [\text{average lives of two types of bulbs are same}]$$

$$H_1: \mu_1 \neq \mu_2 \quad [\text{average lives of two types of bulbs are different}]$$

Since the alternative hypothesis is two-tailed therefore corresponding test will be two-tailed.

If company manager is interested just to know whether average life of CFL is greater than original technology bulbs then our null and alternative hypotheses will be

$$H_0: \mu_1 \geq \mu_2$$

$$H_1: \mu_1 < \mu_2 \quad \left[\begin{array}{l} \text{average life of CFL technology bulbs} \\ \text{is greater than average life of original technology} \end{array} \right]$$

Since the alternative hypothesis is left-tailed therefore the corresponding test will be left-tailed test.

10.3.5 Concept of p-Value

Use of p-value is becoming more and more popular because of the following two reasons:

- most of the statistical software provides p-value rather than the critical value.
- p-value provides more information compared to critical value as far as rejection or does not rejection H_0 .

The first point listed above needs no explanation. But the second point lies in the heart of p-value and needs to explain more clearly. Moving in this direction, we note that in scientific applications one is not only interested simply in rejecting or not rejecting the null hypothesis but he/she is also interested to assess how strong the data has the evidence to reject H_0 . For example, for testing a hypothesis where we test the null hypothesis

$$H_0: \theta \leq 50 \text{ gm against } H_1: \theta > 50 \text{ gm}$$

To test the null hypothesis, we calculated the value of the test statistic as 2.78 and the critical value (z_α) at $\alpha = 0.01$ was $z_\alpha = 2.33$.

Since the calculated value of test statistic (= 2.78) is greater than the critical (tabulated) value (= 2.33), therefore, we reject the null hypothesis at 1% level of significance.

Now, if we reject the null hypothesis at this level (1%) surely we have to reject it at higher level because at $\alpha = 0.05$, $z_\alpha = 1.645$ and at $\alpha = 0.10$, $z_\alpha = 1.28$. However, the calculated value of test statistic is much higher than 1.645 and 1.28, therefore, the question arises “could the null hypothesis also be rejected at values of α smaller than 0.01?” The answer is “yes” and we can compute the smallest level of significance (α) at which a null hypothesis can be rejected. This smallest level of significance (α) is known as “p-value”.

The p-value is the smallest value of the level of significance (α) at which a null hypothesis can be rejected using the obtained value of the test statistic and can be defined as:

The p-value is the probability of obtaining a test statistic equal to or more extreme (in the direction of supporting H_1) than the actual value obtained when null hypothesis is true.

The p-value also depends on the type of test. If a test is one-tailed then the p-value is defined as:

For right-tailed test:

$$\text{p-value} = P[\text{Test Statistic (T)} \geq \text{observed value of the test statistic}]$$

For left-tailed test:

$$\text{p-value} = P[\text{Test Statistic (T)} \leq \text{observed value of the test statistic}]$$

If a test is two-tailed then the p-value is defined as:

For two-tailed test:

$$\text{p-value} = 2P[T \geq |\text{observed value of the test statistic}|]$$

The procedure of taking the decision about the null hypothesis on the basis of p-value:

To take the decision about the null hypothesis based on p-value, the p-value is compared with the level of significance (α) and if p-value is equal or less than

α then we reject the null hypothesis and if the p-value is greater than α we do not reject the null hypothesis.

The p-value for various tests can be obtained with the help of the tables given in the Appendix at the end of this course. But unless we are dealing with the standard normal distribution, the exact p-value is not obtained with the tables as mentioned above. But if we test our hypothesis with the help of computer packages, then these types of computer packages or software present the p-value as part of the output for each hypothesis testing procedure.

Check Your Progress 1

- 1) A company manufactures car tyres. The company claims that the average life of its tyres is 50000 miles. To test the claim of the company, formulate the null and alternative hypotheses.

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- 2) Write the null and alternative hypotheses in cases (iii), (iv) and (v) of example given in Section 10.2.

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- 3) If $H_0: \theta = 60$ and $H_1: \theta \neq 60$ then critical region lies in one-tail or two-tails.

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- 4) If the probability of type-I error is 0.05 then what is the level of significance?

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- 5) If we have null and alternative hypotheses as

$$H_0: \theta = \theta_0 \text{ and } H_1: \theta \neq \theta_0$$

then the corresponding test will be

- (i) left-tailed test (ii) right-tailed test (iii) both-tailed test

Write the correct option.

.....
.....

- 6) The test whether one-tailed or two-tailed depends on

- (i) Null hypothesis (H_0) (ii) Alternative hypothesis (H_1)

- (iii) Neither H_0 nor H_1 (iv) Both H_0 and H_1

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10.4 GENERAL PROCEDURE OF TESTING A HYPOTHESIS

Testing of hypothesis is a huge demanded statistical tool by many discipline and professionals. It is a step by step procedure as you will see in the next three units through a large number of examples. The aim of this section just gives you the flavor of that sequence which involves the following steps:

Step I: First of all, we have to set up the null hypothesis H_0 and alternative hypothesis H_1 . Suppose, we want to test the hypothetical/ claimed/ assumed value θ_0 of parameter θ . So we can take the null and alternative hypotheses as

$$H_0 : \theta = \theta_0 \text{ and } H_1 : \theta \neq \theta_0 \quad [\text{for two-tailed test}]$$

$$\text{or} \quad \left. \begin{array}{l} H_0 : \theta \leq \theta_0 \text{ and } H_1 : \theta > \theta_0 \\ H_0 : \theta \geq \theta_0 \text{ and } H_1 : \theta < \theta_0 \end{array} \right\} \quad [\text{for one-tailed test}]$$

In case of comparing the same parameter of two populations of interest, say, θ_1 and θ_2 , then our null and alternative hypotheses would be

$$H_0 : \theta_1 = \theta_2 \text{ and } H_1 : \theta_1 \neq \theta_2 \quad [\text{for two-tailed test}]$$

$$\text{or} \quad \left. \begin{array}{l} H_0 : \theta_1 \leq \theta_2 \text{ and } H_1 : \theta_1 > \theta_2 \\ H_0 : \theta_1 \geq \theta_2 \text{ and } H_1 : \theta_1 < \theta_2 \end{array} \right\} \quad [\text{for one-tailed test}]$$

Step II: After setting the null and alternative hypotheses, we establish criteria for rejection or non-rejection of the null hypothesis, that is, decide the level of significance (α), at which we want to test our hypothesis. Generally, it is taken as 5% or 1% ($\alpha = 0.05$ or 0.01).

Step III: The third step is to choose an appropriate test statistic under H_0 for testing the null hypothesis, as given below:

$$\text{Test statistic} = \frac{\text{Statistic} - \text{Value of the parameter under } H_0}{\text{Standard error of statistic}}$$

After that, specify the sampling distribution of the test statistic preferably in the standard form like Z (standard normal), χ^2 , t, F or any other well-known in the literature.

Step IV: Calculate the value of the test statistic described in Step III on the basis of observed sample observations.

Step V: Obtain the critical (or cut-off) value(s) in the sampling distribution of the test statistic and construct rejection (critical) region of size α . Generally, critical values for various levels of significance are put in the form of a table for various standard sampling distributions of test statistic such as Z-table, χ^2 -table, t-table, etc. (See Appendix at the end of this course).

Step VI: After that, compare the calculated value of test statistic obtained from Step IV, with the critical value(s) obtained in Step V and locates the position of the calculated test statistic, that is, it lies in rejection region or non-rejection region.

Step VII: In testing of hypothesis ultimately we have to reach at a conclusion. It is done as explained below:

- (i) If the calculated value of test statistic lies in the rejection region at α level of significance then we reject the null hypothesis. It means that the sample data provide us sufficient evidence against the null hypothesis and there is a significant difference between hypothesized value and the observed value of the parameter.

- (ii) If the calculated value of test statistic lies in the non-rejection region at α level of significance then we do not reject the null hypothesis. It means that the sample data fails to provide us sufficient evidence against the null hypothesis and the difference between hypothesized value and the observed value of the parameter due to fluctuation of the sample.

Note 3: Nowadays, the decision about the null hypothesis is taken with the help of p-value. The concept of p-value is very important because computer packages and statistical software all provide p-value.

Now, with the help of an example, we explain the above general procedure.

Example 2: Suppose it is found that the average weight of potatoes was 50 gm and the standard deviation was 5.1 gm nearly 5 years ago. We want to test that due to advancement in agricultural technology, the average weight of potato has been increased. To test this, a random sample of 50 potatoes is taken and calculate the sample mean ($\bar{X}$) as 52 gm. Describe the procedure to carry out this test.

Solution: Here, we are given that

Specified value of population mean = $\mu_0 = 50$ gm,

Population standard deviation = $\sigma = 5.1$ gm,

Sample size = $n = 50$,

Sample mean = $\bar{X} = 52$ gm

To carry out the above test, we have to follow up the following steps:

Step I: First of all, we set up null and alternative hypotheses. Here, we want to test that the average weight of potato is increased. So our claim is “average weight of potato has increased” i.e. $\mu > 50$ and its complement is $\mu \leq 50$. Since complement contains equality sign so we can take the complement as the null hypothesis and claim as the alternative hypothesis, that is,

$$H_0: \mu \leq 50 \text{ gm and } H_1: \mu > 50 \text{ gm [Here, } \theta = \mu]$$

Since the alternative hypothesis is right-tailed, so our test is right-tailed.

Step II: After setting the null and alternative hypotheses, we fix the level of significance α . Suppose, $\alpha = 0.01$ (= 1 % level).

Step III: Define a test statistic to test the null hypothesis as

$$\text{Test statistic} = \frac{\text{Statistic} - \text{Value of the parameter under } H_0}{\text{Standard error of statistic}}$$

$$T = \frac{\bar{X} - 50}{\sigma / \sqrt{n}}$$

Since the sample size is large ($n = 50 > 30$) so by central limit theorem the sampling distribution of test statistic approximately follows standard normal distribution, i.e. $T \sim N(0,1)$

Step IV: Calculate the value of test statistic on the basis of sample observations as

$$T = \frac{52 - 50}{5.1 / \sqrt{50}} = \frac{2}{0.72} = 2.78$$

Step V: Now, we find the critical value. The critical value or cut-off value for standard normal distribution is given in **Table I (Z-table)** in the

Appendix of this course. So from this table, the critical value for right-tailed test at $\alpha = 0.01$ is $z_\alpha = 2.33$.

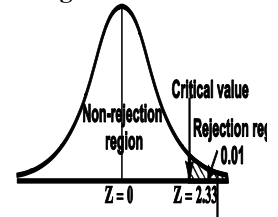
Step VI: Now, to take the decision about the null hypothesis, we compare the calculated value of the test statistic with the critical value.

Since the calculated value of test statistic ($= 2.78$) is greater than the critical value ($= 2.33$), that means the calculated value of test statistic lies in rejection region at 1% level of significance as shown in Fig. 10.4. So we reject the null hypothesis and support the alternative hypothesis. Since the alternative hypothesis is our claim, so we support the claim.

Thus, we conclude that the average weight of potato has increased.

Hypothesis Testing

Fig. 10.4



10.5 TESTING OF HYPOTHESIS FOR POPULATION MEAN USING Z-TEST

In the previous section, we have discussed the general procedure for hypothesis testing. Now we are discussing the Z-test for testing the hypothesis or claim about the population mean when the sample size is large. Let the population under study has mean μ and variance σ^2 , where μ is unknown and σ^2 may be known or unknown. We will consider both cases in this unit. For testing a hypothesis about population mean, we draw a random sample $X_1, X_2, \dots, X_n$ of size $n > 30$ from this population. As we know that for drawing the inference about the population mean, we generally use the sample mean and for the test statistic, we require the mean and standard error of sampling distribution of the statistic (mean). Here, we are considering a large sample so we know by the central limit theorem that sample mean is asymptotically normally distributed with mean μ and variance σ^2/n whether parent population is **normal or non-normal**. That is, if $\bar{X}$ is the sample mean of the random sample then

$$E(\bar{X}) = \mu, \text{ Var}(\bar{X}) = \frac{\sigma^2}{n} \quad \dots (1)$$

But we know that standard error = $\sqrt{\text{Variance}}$

$$\therefore SE(\bar{X}) = \sqrt{\text{Var}(\bar{X})} = \frac{\sigma}{\sqrt{n}} \quad \dots (2)$$

Now, follow the same procedure as we have discussed in the previous section, that is, first of all, we have to set up null and alternative hypotheses. Since here we want to test the hypothesis about the population mean so we can take the null and alternative hypotheses as

$$\begin{aligned} & H_0 : \mu = \mu_0 \text{ and } H_1 : \mu \neq \mu_0 \text{ [for two-tailed test] } \left[\begin{array}{l} \text{Here, } \theta = \mu \text{ and } \theta_0 = \mu_0 \\ \text{if we compare it with} \\ \text{general procedure.} \end{array} \right] \\ \text{or} \quad & \left. \begin{array}{l} H_0 : \mu \leq \mu_0 \text{ and } H_1 : \mu > \mu_0 \\ H_0 : \mu \geq \mu_0 \text{ and } H_1 : \mu < \mu_0 \end{array} \right\} \text{ [for one-tailed test]} \end{aligned}$$

When we assume that the null hypothesis is true then we are actually assuming that the population parameter is equal to the value in the null hypothesis. is $\mu = 60$ or $\mu \leq 60$ or $\mu \geq 60$.

For testing the null hypothesis, the test statistic Z is given by

$$Z = \frac{\bar{X} - E(\bar{X})}{SE(\bar{X})}$$

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \quad \left[\begin{array}{l} \text{Using equations (1) and (2) and} \\ \text{under } H_0 \text{ we assume that } \mu = \mu_0. \end{array} \right]$$

The sampling distribution of the test statistic depends upon σ^2 that it is known or unknown. Therefore, two cases arise:

Case I: When σ^2 is known

In this case, the test statistic follows the normal distribution with mean 0 and variance unity when the sample size is large as the population under study is normal or non-normal. If the sample size is small then the test statistic Z follows the normal distribution only when the population under study is normal. Thus,

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0, 1)$$

Case II: When σ^2 is unknown

In this case, we estimate σ^2 by the value of sample variance (S^2) where,

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

Then the test statistic becomes $\frac{\bar{X} - \mu_0}{S/\sqrt{n}}$ and follows the t-distribution

with $(n-1)$ degrees of freedom (df) as the sample size is large or small, provided the population under study is normal. But when the population under study is not normal and the sample size is large then this test statistic approximately follows a normal distribution with mean 0 and variance unity, that is,

$$Z = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim N(0, 1)$$

After that, we calculate the value of the test statistic as may be the case (σ^2 is known or unknown) and compare it with the critical value given in **Tables of Z-value given in the Appendix at the end of this course**, at the prefixed level of significance α . Take the decision about the null hypothesis as described in the previous section.

From the above discussion of testing of hypothesis about the population mean, we note the following point:

- (i) When σ^2 is known then we apply the Z-test as the population under study is normal or non-normal for the large sample. But when the sample size is small then we apply the Z-test only when the population under study is normal.
- (ii) When σ^2 is unknown then we apply the t-test only when the population under study is normal as the sample size is large or small. But when the assumption of normality is not fulfilled and the sample size is large then we can apply the Z-test.
- (iii) When the sample is small and σ^2 is known or unknown and the form of the population is not known then we apply the non-parametric test, which are not part of this course.

The following examples will help you to understand the procedure more clearly.

Example 3: A light bulb company claims that the 100-watt light bulb it sells has an average life of 1200 hours with a standard deviation of 100 hours. For testing the claim 50 new bulbs were selected randomly and allowed to burn out. The average lifetime of these bulbs was found to be 1180 hours. Is the company's claim true at 5% level of significance?

Solution: Here, we are given that

Specified value of population mean = $\mu_0 = 1200$ hours,

Population standard deviation = $\sigma = 100$ hours,

Sample size = $n = 50$

Sample mean = $\bar{X} = 1180$ hours.

In this example, the population parameter being tested is the population mean i.e. the average life of a bulb (μ) and we want to test the company's claim that the average life of a bulb is 1200 hours. So our claim is $\mu = 1200$ and its complement is $\mu \neq 1200$. Since the claim contains the equality sign so we can take the claim as the null hypothesis and complement as the alternative hypothesis. So

$$H_0 : \mu = \mu_0 = 1200 \text{ [average life of the bulbs is 1200 hours]}$$

$$H_1 : \mu \neq 1200 \text{ [average life of the bulbs is not 1200 hours]}$$

Also, the alternative hypothesis is two-tailed, so the test is two-tailed.

Here, we want to test the hypothesis regarding mean when population SD (variance) is known and sample size $n = 50 (> 30)$ is large. So we will go for Z-test.

Thus, for testing the null hypothesis the test statistic is given by

$$\begin{aligned} Z &= \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}} \\ &= \frac{1180 - 1200}{100 / \sqrt{50}} = \frac{-20}{14.14} = -1.41 \end{aligned}$$

The critical (tabulated) values for two-tailed test at 5% level of significance are $\pm Z_{\alpha/2} = \pm Z_{0.025} = \pm 1.96$.

Since the calculated value of the test statistic $Z (= -1.41)$ is greater than the critical value ($= -1.96$) and less than the critical value ($= 1.96$), that means it lies in the non-rejection region as shown in Fig. 10.5, so we do not reject the null hypothesis. Since the null hypothesis is the claim so we support the claim at 5% level of significance.

Decision according to p-value:

The test is two-tailed, therefore,

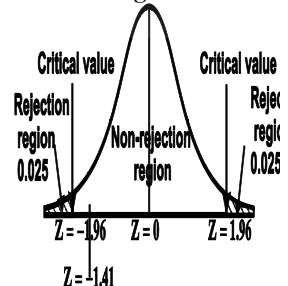
$$p\text{-value} = 2P[Z \geq |z|] = 2P[Z \geq 1.41]$$

$$= 2[0.5 - P[0 \leq Z \leq 1.41]] = 2(0.5 - 0.4207) = 0.1586$$

Since p-value ($= 0.1586$) is greater than $\alpha (= 0.05)$ so we do not reject the null hypothesis at 5% level of significance.

Decision according to confidence interval:

Fig. 10.5



Here, the test is two-tailed, therefore, we construct a two-sided confidence interval for the population mean.

Since population standard deviation is known, therefore, we can use $(1-\alpha)$ 100 % confidence interval for the population mean when the population variance is known which is given by

$$\left[\bar{X} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}}, \bar{X} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \right]$$

Here, $\alpha = 0.05$, we have $z_{\alpha/2} = z_{0.025} = 1.96$.

Thus, 95% confidence interval for the average life of a bulb is given by

$$\left[1180 - 1.96 \frac{100}{\sqrt{50}}, 1180 + 1.96 \frac{100}{\sqrt{50}} \right]$$

or $[1180 - 27.71, 1180 + 27.71]$

or $[1152.29, 1207.71]$

Since 95% confidence interval for the average life of a bulb contains the value of the parameter specified by the null hypothesis, that is, $\mu = \mu_0 = 1200$ so we do not reject the null hypothesis.

Thus, we conclude that the sample does not provide us sufficient evidence against the claim so we may assume that the company's claim that the average life of a bulb is 1200 hours is true.

Note 4: Here, we note that the decisions about the null hypothesis based on three approaches (critical value or classical p-value and confidence interval) are same. The learners are advised to make the decision about the claim or statement by using only one of the three approaches in the examination. Here, we used all these approaches only to give you an idea how they can be used in a given problem.

Example 4: A manufacturer of ballpoint pens claims that a certain pen manufactured by him has a mean writing life at least 460 A-4 size pages. A purchasing agent selects a sample of 100 pens and put them on the test. The mean writing-life of the sample found 453 A-4 size pages with standard deviation 25 A-4 size pages. Should the purchasing agent reject the manufacturer's claim at 1% level of significance?

Solution: Here, we are given that

Specified value of population mean $= \mu_0 = 460$,

Sample size $= n = 100$,

Sample mean $= \bar{X} = 453$,

Sample standard deviation $= S = 25$

Here, we want to test the manufacturer's claim that the mean writing-life (μ) of the pen is at least 460 A-4 size pages. So our claim is $\mu \geq 460$ and its complement is $\mu < 460$. Since the claim contains the equality sign so we can take the claim as the null hypothesis and the complement as the alternative hypothesis. So

$$H_0 : \mu \geq \mu_0 = 460 \text{ and } H_1 : \mu < 460$$

Also, the alternative hypothesis is left-tailed so the test is left-tailed.

Here, we want to test the hypothesis regarding population mean when population SD is unknown. So we should use the t-test if the writing life of the pen follows a normal distribution. But it is not the case. Since the sample size is $n = 100$ ($n > 30$) large so we go for the Z-test. The test statistic of the Z-test is given by

$$Z = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$$

$$= \frac{453 - 460}{25/\sqrt{100}} = \frac{-7}{2.5} = -2.8$$

From Table I of the Appendix given at the end of this course, the value for the left-tailed Z test at 1% level of significance is $z_\alpha = -2.33$.

Since the calculated value of the test statistic Z ($= -2.8$) is less than the critical value ($= -2.33$), that means the calculated value of test statistic Z lies in the rejection region as shown in Fig. 10.6, so we reject the null hypothesis. Since the null hypothesis is the claim so we reject the manufacturer's claim at 1% level of significance.

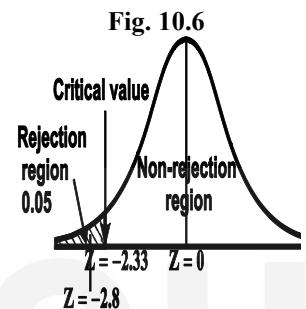
Decision according to p-value:

The test is left-tailed, therefore,

$$\begin{aligned} \text{p-value} &= P[Z \leq z] = P[Z \leq -2.8] = P[Z \geq 2.8] \quad \left[\because Z \text{ is symmetrical about } Z = 0 \text{ line} \right] \\ &= 0.5 - P[0 \leq Z \leq 2.8] = 0.5 - 0.4974 = 0.0026 \end{aligned}$$

Since the p-value ($= 0.0026$) is less than α ($= 0.01$) so we reject the null hypothesis at 1% level of significance.

Therefore, we conclude that the sample provides us sufficient evidence against the claim so the purchasing agent rejects the manufacturer's claim at 1% level of significance.



10.6 TESTING OF HYPOTHESIS FOR DIFFERENCE OF TWO POPULATION MEANS USING Z-TEST

In the previous section, we have learnt about the testing of hypothesis about the population mean. But there are so many situations where we want to test the hypothesis about difference of two population means or two population means. For example, two manufacturing companies of bulbs produce the same type of bulbs and one may be interested to test that one is better than the other, an investigator may want to test the equality of the average incomes of the people living in two cities, etc. Therefore, we require an appropriate test for testing the hypothesis about the difference of two population means.

Let there be two populations, say, population-I and population-II under study. Also, let μ_1, μ_2 and σ_1^2, σ_2^2 denote the means and variances of population-I and population-II respectively where both μ_1 and μ_2 are unknown but σ_1^2 and σ_2^2 may be known or unknown. We will consider all

possible cases here. For testing the hypothesis about the difference of two population means, we draw a random sample of large size n_1 from population-I and a random sample of large size n_2 from population-II. Let $\bar{X}$ and $\bar{Y}$ be the means of the samples selected from population-I and II, respectively.

These two populations may or may not be normal but according to the central limit theorem, the sampling distribution of difference of two large sample means asymptotically normally distributed with mean $(\mu_1 - \mu_2)$ and variance $(\sigma_1^2/n_1 + \sigma_2^2/n_2)$ as described earlier in this course.

Thus,

$$E(\bar{X} - \bar{Y}) = E(\bar{X}) - E(\bar{Y}) = \mu_1 - \mu_2 \quad \dots (3)$$

and

$$\text{Var}(\bar{X} - \bar{Y}) = \text{Var}(\bar{X}) + \text{Var}(\bar{Y}) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

But we know that standard error = $\sqrt{\text{Variance}}$

$$\therefore SE(\bar{X} - \bar{Y}) = \sqrt{\text{Var}(\bar{X} - \bar{Y})} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \quad \dots (4)$$

Now, follow the same procedure as we have discussed in previous section, that is, first of all, we have to set up null and alternative hypotheses. Here, we want to test the hypothesis about the difference of two population means so we can take the null hypothesis as

$$H_0 : \mu_1 = \mu_2 \text{ (no difference in means)}$$

$$\text{or } H_0 : \mu_1 - \mu_2 = 0 \text{ (difference in two means is 0)}$$

and the alternative hypothesis as

$$H_1 : \mu_1 \neq \mu_2 \quad [\text{for two-tailed test}]$$

$$\text{or } \left. \begin{array}{l} H_0 : \mu_1 \leq \mu_2 \text{ and } H_1 : \mu_1 > \mu_2 \\ H_0 : \mu_1 \geq \mu_2 \text{ and } H_1 : \mu_1 < \mu_2 \end{array} \right\} [\text{for one-tailed test}]$$

For testing the null hypothesis, the test statistic Z is given by

$$Z = \frac{(\bar{X} - \bar{Y}) - E(\bar{X} - \bar{Y})}{SE(\bar{X} - \bar{Y})}$$

$$\text{or } Z = \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \quad [\text{using equations (3) and (4)}]$$

Since under the null hypothesis, we assume that $\mu_1 = \mu_2$, therefore, we have

$$Z = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

Now, the sampling distribution of the test statistic depends upon σ_1^2 and σ_2^2 that both are known or unknown. Therefore, four cases arise:

Case I: When σ_1^2 & σ_2^2 are known and $\sigma_1^2 = \sigma_2^2 = \sigma^2$

In this case, the test statistic follows a normal distribution with mean 0 and variance unity when the sample sizes are large as both the populations under study are normal or non-normal. But when sample sizes are small then the test statistic Z follows normal distribution only when populations under study are normal, that is,

$$Z = \frac{\bar{X} - \bar{Y}}{\sigma \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim N(0,1)$$

Case II: When σ_1^2 & σ_2^2 are known and $\sigma_1^2 \neq \sigma_2^2$

In this case, the test statistic also follows the normal distribution as described in case I, that is,

$$Z = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1)$$

Case III: When σ_1^2 & σ_2^2 are unknown and $\sigma_1^2 = \sigma_2^2 = \sigma^2$

In this case, σ^2 is estimated by the value of pooled sample variance S_p^2 where,

$$S_p^2 = \frac{1}{n_1 + n_2 - 2} [n_1 S_1^2 + n_2 S_2^2]$$

and

$$S_1^2 = \frac{1}{(n_1 - 1)} \sum_{i=1}^{n_1} (X_i - \bar{X})^2 \text{ and } S_2^2 = \frac{1}{(n_2 - 1)} \sum_{i=1}^{n_2} (Y_i - \bar{Y})^2$$

and the test statistic follows t-distribution with $(n_1 + n_2 - 2)$ degrees of freedom as the sample sizes are large or small provided populations under study follow a normal distribution. But when the populations are under study are not normal and sample sizes n_1 and n_2 are large (> 30) then by central limit theorem, the test statistic approximately normally distributed with mean 0 and variance unity, that is,

$$Z = \frac{\bar{X} - \bar{Y}}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim N(0,1)$$

Case IV: When σ_1^2 & σ_2^2 are unknown and $\sigma_1^2 \neq \sigma_2^2$

In this case, σ_1^2 & σ_2^2 are estimated by the values of the sample variances S_1^2 & S_2^2 respectively and the exact distribution of the test statistic is difficult to derive. But when sample sizes n_1 and n_2 are large (> 30) then according to the central limit theorem, the test statistic is approximately normally distributed with mean 0 and variance unity, that is,

$$Z = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} \sim N(0, 1)$$

After that, we calculate the value of the test statistic and compare it with the critical value given in the Appendix at a prefixed level of significance α . Take the decision about the null hypothesis as described in the last section.

From the above discussion of testing of hypothesis about population mean, we note the following point:

- (i) When σ_1^2 & σ_2^2 are known then we apply the Z-test as both the population under study are normal or non-normal for the large sample. But when sample sizes are small then we apply the Z-test only when populations under study are normal.
- (ii) When σ_1^2 & σ_2^2 are unknown then we apply the t-test only when the populations under study are normal as sample sizes are large or small. But when the assumption of normality is not fulfilled and sample sizes are large then we can apply the Z-test.
- (iii) When samples are small and σ_1^2 & σ_2^2 are known or unknown and the form of the population is not known then we apply the non-parametric tests which are beyond the scope of this course.

Let us do some examples based on the above test.

Example 5: In two samples of women from Punjab and Tamil Nadu, the mean height of 1000 and 2000 women are 67.6 and 68.0 inches respectively. If the population standard deviation of Punjab and Tamil Nadu are same and equal to 5.5 inches then, can the mean heights of Punjab and Tamil Nadu women be regarded as same at 1% level of significance?

Solution: We are given

$$n_1 = 1000, n_2 = 2000, \bar{X} = 67.6, \bar{Y} = 68.0 \text{ and } \sigma_1 = \sigma_2 = \sigma = 5.5$$

Here, we wish to test that the mean height of Punjab and Tamil Nadu women is same. If μ_1 and μ_2 denote the mean heights of Punjab and Tamil Nadu women respectively then our claim is $\mu_1 = \mu_2$ and its complement is $\mu_1 \neq \mu_2$. Since the claim contains the equality sign so we can take the claim as the null hypothesis and complement as the alternative hypothesis. Thus,

$$H_0 : \mu_1 = \mu_2 \text{ and } H_1 : \mu_1 \neq \mu_2$$

Since the alternative hypothesis is two-tailed so the test is two-tailed.

Here, we want to test the hypothesis regarding two population means. The standard deviations of both populations are known and sample sizes are large, so we should go for Z-test.

So, for testing the null hypothesis, the test statistic Z is given by

$$Z = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

$$\begin{aligned}
 &= \frac{67.6 - 68.0}{\sqrt{\left(\frac{(5.5)^2}{1000} + \frac{(5.5)^2}{2000}\right)}} = \frac{-0.4}{5.5 \sqrt{\left(\frac{1}{1000} + \frac{1}{2000}\right)}} \\
 &= \frac{-0.4}{5.5 \times 0.0387} = -1.88
 \end{aligned}$$

The critical (tabulated) values for the two-tailed test at 1% level of significance are $\pm z_{\alpha/2} = \pm z_{0.005} = \pm 2.58$.

Since the calculated value of Z ($= -1.88$) is greater than the critical value ($= -2.58$) and less than the critical value ($= 2.58$), that means it lies in the non-rejection region as shown in Fig. 10.7, so we do not reject the null hypothesis i.e. we fail to reject the claim.

Decision according to p-value:

The test is two-tailed, therefore,

$$\begin{aligned}
 \text{p-value} &= 2P[Z \geq |z|] = 2P[Z \geq 1.88] \\
 &= 2[0.5 - P[0 \leq Z \leq 1.88]] = 2(0.5 - 0.4699) = 0.0602
 \end{aligned}$$

Since the p-value ($= 0.0602$) is greater than $\alpha (= 0.01)$ so we do not reject the null hypothesis at 1% level of significance.

Thus, we conclude that the samples do not provide us sufficient evidence against the claim so we may assume that the average height of women of Punjab and Tamil Nadu is the same.

Example 6: A university conducts both face to face and distance mode classes for a particular course intended to be identical. A sample of 50 students of face to face mode yields examination results mean and SD respectively as:

$$\bar{X} = 80.4, \quad S_1 = 12.8$$

and another sample of 100 distance-mode students yields mean and SD of their examination results in the same course respectively as:

$$\bar{Y} = 74.3, \quad S_2 = 20.5$$

Are both educational methods statistically equal at 5% level?

Solution: Here, we are given that

$$n_1 = 50, \quad \bar{X} = 80.4, \quad S_1 = 12.8;$$

$$n_2 = 100, \quad \bar{Y} = 74.3, \quad S_2 = 20.5$$

We wish to test that both educational methods are statistically equal. If μ_1 and μ_2 denote the average marks of face to face and distance mode students respectively then our claim is $\mu_1 = \mu_2$ and its complement is $\mu_1 \neq \mu_2$. Since the claim contains the equality sign so we can take the claim as the null hypothesis and complement as the alternative hypothesis. Thus,

$$H_0 : \mu_1 = \mu_2 \text{ and } H_1 : \mu_1 \neq \mu_2$$

Since the alternative hypothesis is two-tailed so the test is two-tailed.

We want to test the null hypothesis regarding two population means when standard deviations of both populations are unknown. So we should go for

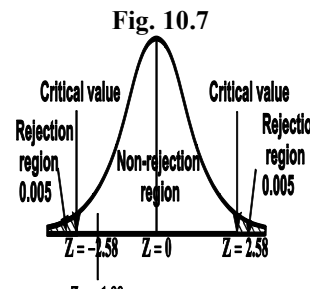
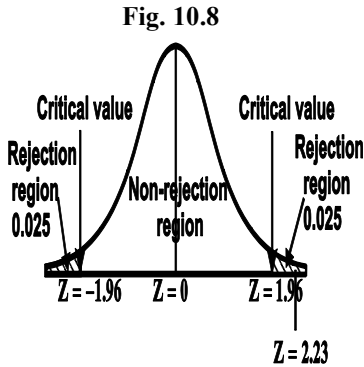


Fig. 10.7

the t-test if the population of difference is known to be normal. But it is not the case. Since sample sizes are large (n_1 , and $n_2 > 30$) so we go for Z-test.

For testing the null hypothesis, the test statistic Z is given by

$$Z = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} = \frac{80.4 - 74.3}{\sqrt{\frac{(12.8)^2}{50} + \frac{(20.5)^2}{100}}} = \frac{6.1}{\sqrt{3.28 + 4.20}} = 2.23$$



The critical (tabulated) values for the two-tailed test at 5% level of significance are $\pm Z_{\alpha/2} = \pm Z_{0.025} = \pm 1.96$.

Since the calculated value of Z (= 2.23) is greater than the critical values (= ± 1.96), that means it lies in the rejection region as shown in Fig. 10.8, so we reject the null hypothesis i.e. we reject the claim at 5% level of significance.

Decision according to p-value:

The test is two-tailed, therefore,

$$\begin{aligned} \text{p-value} &= 2P[Z \geq |z|] = 2P[Z \geq 2.23] \\ &= 2[0.5 - P[0 \leq Z \leq 2.23]] = 2(0.5 - 0.4871) = 0.0258 \end{aligned}$$

Since the p-value (= 0.0258) is less than α (= 0.05) so we reject the null hypothesis at 5% level of significance.

Thus, we conclude that the samples provide us sufficient evidence against the claim so both methods of education, i.e. face-to-face and distance-mode, provided by that University are not statistically equal.

10.7 TESTING OF HYPOTHESIS FOR POPULATION PROPORTION USING Z-TEST

In the last section, we have discussed the procedure of testing of hypothesis for the population mean when the sample size is large and the data is quantitative. But in many real-world situations, in business and other areas where collected data are in the form of counts or the collected data are classified into two categories or groups according to an attribute or a characteristic. For example, the people living in a colony may be classified into two groups (male and female) with respect to the characteristic gender, the patients in a hospital may be classified into two groups as cancer and non-cancer patients, articles may be classified as defective and non-defective, etc. Here, collected data are available in dichotomous or binary outcomes which is a special case of nominal scale and the data is categorised into two mutually exclusive and exhaustive classes generally known as success and failure outcomes. For example, the characteristic gender can be measured as a success if female and a failure if male or vice versa. So in such situations, proportion is a suitable measure to apply.

In such situations, we require a test for testing a hypothesis about the population proportion.

For this purpose, let $X_1, X_2, \dots, X_n$ be a random sample of size n taken from a population with population proportion P . Also let X denotes the number of observations or elements that possess a certain attribute (number of successes) out of n observations of the sample then sample proportion p can be defined as

$$p = \frac{X}{n} \leq 1$$

As we have seen in an earlier unit of this course that the mean and variance of the sampling distribution of sample proportion are:

$$E(p) = P \text{ and } \text{Var}(p) = \frac{PQ}{n}$$

where $Q = 1 - P$.

Now, two cases arise:

Case I: When the sample size is not sufficiently large i.e. either of the conditions $np > 5$ or $nq > 5$ does not meet, then we use the exact binomial test. But the exact binomial test is beyond the scope of this course.

Case II: When sample size is sufficiently large, such that $np > 5$ and $nq > 5$ then by central limit theorem, the sampling distribution of sample proportion p is approximately normally distributed with mean and variance as

$$E(p) = P \text{ and } \text{Var}(p) = \frac{PQ}{n} \quad \dots (5)$$

But we know that standard error = $\sqrt{\text{Variance}}$

$$\therefore SE(p) = \sqrt{\frac{PQ}{n}} \quad \dots (6)$$

Now, follow the same procedure as we have discussed in the last Section. First of all, we set up null and alternative hypotheses. Since here we want to test the hypothesis about a specified value P_0 of the population proportion so we can take the null and alternative hypotheses as

$$H_0 : P = P_0 \text{ and } H_1 : P \neq P_0 \text{ [for two-tailed test]} \left[\begin{array}{l} \text{Here, } \theta = P \text{ and } \theta_0 = P_0 \\ \text{if we compare it with} \\ \text{general procedure.} \end{array} \right]$$

$$\text{or } \left. \begin{array}{l} H_0 : P \leq P_0 \text{ and } H_1 : P > P_0 \\ H_0 : P \geq P_0 \text{ and } H_1 : P < P_0 \end{array} \right\} \text{ [for one-tailed test]}$$

For testing the null hypothesis, the test statistic Z is given by

$$Z = \frac{p - E(p)}{SE(p)}$$

$$Z = \frac{p - P_0}{\sqrt{\frac{P_0 Q_0}{n}}} \sim N(0, 1) \text{ under } H_0 \text{ [using equations (5) and (6)]}$$

After that, we calculate the value of the test statistic and compare it with the critical value(s) at a prefixed level of significance α . We take the decision about the null hypothesis as described in the last Section.

Let us do some examples of testing of hypothesis about population proportion.

Example 7: A machine produces a large number of items out of which 25% are found to be defective. To check this, the company manager takes a random sample of 100 items and found 35 items defective. Is there evidence of more deterioration of quality at 5% level of significance?

Solution: The company manager wants to check that his machine produces 25% defective items. Here, attribute under study is defectiveness. And we define our success and failure as getting a defective or non-defective item.

Let P = Population proportion of defectives items = 0.25 ($= P_0$)

p = Observed proportion of defectives items in the sample = $35/100 = 0.35$

Here, we want to test that the machine produces more defective items, that is, the proportion of defective items (P) greater than 0.25. So our claim is $P > 0.25$ and its complement is $P \leq 0.25$. Since complement contains the equality sign so we can take the complement as the null hypothesis and the claim as the alternative hypothesis. So

$$H_0 : P \leq P_0 = 0.25 \text{ and } H_1 : P > 0.25$$

Since the alternative hypothesis is right-tailed so the test is right-tailed.

Before proceeding further, first, we have to check whether the condition of normality meets or not.

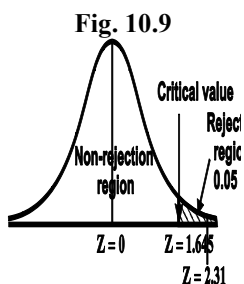
$$\therefore np = 100 \times 0.35 = 35 > 5$$

$$nq = 100 \times (1 - 0.35) = 100 \times 0.65 = 65 > 5$$

We see that condition of normality meets, so we can go for Z-test.

So, for testing the null hypothesis, the test statistic Z is given by

$$\begin{aligned} Z &= \frac{p - P_0}{\sqrt{\frac{P_0 Q_0}{n}}} \\ &= \frac{0.35 - 0.25}{\sqrt{\frac{0.25 \times 0.75}{100}}} = \frac{0.10}{0.0433} = 2.31 \end{aligned}$$



Since test is right-tailed so the critical value at 5% level of significance is $Z_{\alpha} = Z_{0.05} = 1.645$.

Since the calculated value of the test statistic Z ($= 2.31$) is greater than the critical value ($= 1.645$), that means it lies in the rejection region as shown in Fig. 10.9. So we reject the null hypothesis and support the alternative hypothesis i.e. we support the claim at 5% level of significance.

Decision according to p-value:

The test is right-tailed, therefore,

$$\begin{aligned}
 \text{p-value} &= P[Z \geq z] = P[Z \geq 2.31] \\
 &= 0.5 - P[0 < Z < 2.31] = 0.5 - 0.4896 \\
 &= 0.0104
 \end{aligned}$$

Since the p-value (= 0.0104) is less than α (= 0.05) so we reject the null hypothesis at 5% level of significance.

Thus, we conclude that the sample fails to provide us sufficient evidence against the claim so we may assume that deterioration in quality exists at 5% level of significance.

Example 8: A die is thrown 9000 times and a draw of 2 or 5 is observed 3100 times. Can we regard that die is unbiased at 5% level of significance?

Solution: Let getting a 2 or 5 be our success, and getting a number other than 2 or 5 to be a failure. In usual notions, we have

$$n = 9000, X = \text{number of successes} = 3100, p = 3100/9000 = 0.3444$$

Here, we want to test that the die is unbiased and we know that if die is unbiased then the proportion or probability of getting 2 or 5 is

$$\begin{aligned}
 P &= \text{Probability of getting a 2 or 5} \\
 &= \text{Probability of getting 2} + \text{Probability of getting 5} \\
 &= \frac{1}{6} + \frac{1}{6} = \frac{1}{3} = 0.3333
 \end{aligned}$$

So our claim is $P = 0.3333$ and its complement is $P \neq 0.3333$. Since the claim contains the equality sign so we can take the claim as the null hypothesis and complement as the alternative hypothesis. Thus,

$$H_0 : P = P_0 = 0.3333 \quad \text{and} \quad H_1 : P \neq 0.3333$$

Since the alternative hypothesis is two-tailed so the test is two-tailed.

Before proceeding further, first, we have to check whether the condition of normality meets or not.

$$\therefore np = 9000 \times 0.3444 = 3099.6 > 5$$

$$nq = 9000 \times (1 - 0.3444) = 9000 \times 0.6556 = 5900.4 > 5$$

We see that condition of normality meets, so we can go for Z-test.

So, for testing the null hypothesis, the test statistic Z is given by

$$\begin{aligned}
 Z &= \frac{p - P_0}{\sqrt{\frac{P_0 Q_0}{n}}} \\
 &= \frac{0.3444 - 0.3333}{\sqrt{\frac{0.3333 \times 0.6667}{9000}}} = \frac{0.0111}{0.005} = 2.22
 \end{aligned}$$

Since the test is a two-tailed test so the critical values at 5% level of significance are $\pm Z_{\alpha/2} = \pm Z_{0.025} = \pm 1.96$.

Since the calculated value of Z (= 2.22) is greater than the critical value (= 1.96), that means it lies in the rejection region, so we reject the null hypothesis i.e. we reject our claim.

Decision according to p-value:

Since the test is two-tailed so

$$\begin{aligned} \text{p-value} &= 2P[Z \geq |z|] = 2P[Z \geq 2.22] \\ &= 2[0.5 - P[0 < Z < 2.22]] = 2(0.5 - 0.4868) = 0.0264 \end{aligned}$$

Since the p-value (= 0.0264) is less than α (= 0.05), so we reject the null hypothesis at 5% level of significance.

Thus, we conclude that the sample provides us sufficient evidence against the claim so die cannot be considered as unbiased.

Check Your Progress 2

- 1) A sample of 900 bolts has a mean length 3.4 cm. Is the sample regarded to be taken from a large population of bolts with mean length 3.25 cm and standard deviation 2.61 cm at 5% level of significance?

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.....

- 2) Two brands of electric bulbs are quoted at the same price. A buyer has tested a random sample of 200 bulbs of each brand and found the following information:

	Mean Life (hrs.)	SD(hrs.)
Brand A	1300	41
Brand B	1280	46

Is there any significant difference in the mean duration of their lives of two brands of electric bulbs at 1% level of significance?

.....
.....

- 3) Out of 200 patients who are given a particular injection 180 survived. Test the hypothesis that the survival rate is more than 80% at 5% level of significance?

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10.8 TESTING OF HYPOTHESIS FOR POPULATION MEAN USING t-TEST

In the last Section, we have discussed Z-test for testing the hypothesis about the population mean when population variance σ^2 is known and unknown.

Recall from these, we have already pointed out that one basic difference between the Z-test and the t-test is that the Z-test is used when population SD is known whether the sample size is large or small and t-test is used when population SD is unknown whether the sample size is small or large. But in the case of large sample size, the Z-test is appropriate for the t-test. But in practice, the standard deviation of the population is not known and the sample size is small so in this situation, we use the t-test provided population under study is normal.

Assumptions

Virtually every test has some assumptions which must be met before the application of the test. This t-test needs the following assumptions to work:

- (i) The characteristic under study follows a normal distribution. In other words, populations from which a random sample is drawn should be normal with respect to the characteristic of interest.
- (ii) Sample observations are random and independent.
- (iii) Population variance σ^2 is unknown.

For describing this test, let $X_1, X_2, \dots, X_n$ be a random sample of **small size n (< 30)** selected from a **normal population** with mean μ and unknown variance σ^2 .

Now, follow the same procedure as we have discussed in the previous section, that is, first of all, we set up the null and alternative hypotheses. Here, we want to test the claim about the specified value μ_0 of the population mean μ so we can take the null and alternative hypotheses as

$$\begin{aligned}
 &H_0 : \mu = \mu_0 \text{ and } H_1 : \mu \neq \mu_0 \quad \left[\text{for two-tailed test} \right] \quad \left[\begin{array}{l} \text{Here, } \theta = \mu \text{ and } \theta_0 = \mu_0 \\ \text{if we compare it with} \\ \text{general procedure.} \end{array} \right] \\
 \text{or} \quad &\left. \begin{aligned} H_0 : \mu \leq \mu_0 \text{ and } H_1 : \mu > \mu_0 \\ H_0 : \mu \geq \mu_0 \text{ and } H_1 : \mu < \mu_0 \end{aligned} \right\} \quad \left[\text{for one-tailed test} \right]
 \end{aligned}$$

For testing the null hypothesis, the test statistic t is given by

$$t = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim t_{(n-1)} \quad \text{under } H_0$$

where, $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$ is the sample mean and $S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$ is the sample variance.

For computational simplicity, we may use the following formulae for $\bar{X}$, S^2 :

$$\bar{X} = a + \frac{1}{n} \sum d \text{ and } S^2 = \frac{1}{n-1} \left[\sum d^2 - \frac{(\sum d)^2}{n} \right]$$

where, $d = (X - a)$, 'a' being the assumed arbitrary value.

Here, the test statistic t follows t-distribution with $(n - 1)$ degrees of freedom.

After substituting values of $\bar{X}$, S and n , we get the calculated value of the test statistic t . Then we look for critical (or cut-off or tabulated) value(s) of the test statistic t from the t-table, given in the Appendix at the end of this course. On comparing calculated value and critical value(s), we take the decision about the null hypothesis as discussed in the previous section.

Let us do some examples of testing of hypothesis about the population mean using the t-test.

Example 9: A manufacturer claims that a special type of projector bulb has an average life 160 hours. To check this claim an investigator takes a sample of 20 such bulbs, puts on the test, and obtains an average life 167 hours with standard deviation 16 hours. Assuming that the lifetime of such bulbs follows a normal

distribution, does the investigator accept the manufacturer's claim at 5% level of significance?

Solution: Here, we are given that

$$\mu_0 = 160, \quad n = 20, \quad \bar{X} = 167 \quad \text{and} \quad S = 16$$

Here, we want to test the manufacturer's claims that a special type of projector bulb has an average life (μ) 160 hours. So claim is $\mu = 160$ and its complement is $\mu \neq 160$. Since the claim contains the equality sign so we can take the claim as the null hypothesis and complement as the alternative hypothesis. Thus,

$$H_0 : \mu = \mu_0 = 160 \quad \text{and} \quad H_1 : \mu \neq 160$$

Since the alternative hypothesis is two-tailed so the test is two-tailed.

Here, we want to test the hypothesis regarding population mean when population SD is unknown. Also, the sample size is small $n = 20$ ($n < 30$) and the population under study is normal, so we can go for the t-test for testing the hypothesis about the population mean.

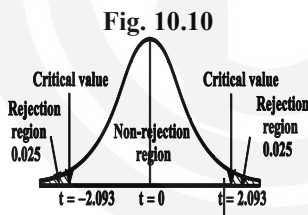
For testing the null hypothesis, the test statistic t is given by

$$\begin{aligned} t &= \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \\ &= \frac{167 - 160}{16/\sqrt{20}} = \frac{7}{3.58} = 1.96 \end{aligned}$$

The critical value of the test statistic t for various df and different level of significance α are given in Table V of the Appendix at the end of this course.

The critical (tabulated) values of the test statistic for two-tailed test corresponding $(n-1) = 19$ df at 5% level of significance are

$$\pm t_{(n-1), \alpha/2} = \pm t_{(19), 0.025} = \pm 2.093.$$



Since the calculated value of the test statistic t ($= 1.96$) is greater than the critical value ($= -2.093$) and is less than the critical value ($= 2.093$), that means the calculated value of the test statistic lies in the non-rejection region as shown in Fig. 10.10. So we do not reject the null hypothesis i.e. we support the manufacturer's claim at 5% level of significance.

Decision according to p-value:

Since the calculated value of the test statistic t is based on 19 df , therefore, we use row for 19 df in the t -table (Table V in Appendix) and move across this row to find the values in which the calculated t -value falls. Since the calculated t -value falls between 1.729 and 2.093 corresponding to one-tailed area $\alpha = 0.05$ and 0.025 respectively therefore p -value lies between 0.025 and 0.05, that is,

$$0.025 < p\text{-value} < 0.05$$

Since the test is two-tailed so

$$2 \times 0.025 = 0.05 < p\text{-value} < 0.10 = 0.05 \times 2$$

Since the p -value is greater than α ($= 0.05$) so we do not reject the null hypothesis at 5% level of significance.

Thus, we conclude that the sample fails to provide us sufficient evidence against the null hypothesis so we may assume that the manufacturer's claim is true so the investigator may accept the manufacturer's claim at 5% level of significance.

Example 10: The mean share price of companies in the Pharma sector is Rs.70. The share prices of all companies were changed from time to time. After a month, a sample of 10 Pharma companies was taken and their share prices were noted as below:

70, 76, 75, 69, 70, 72, 68, 65, 75, 72

Assuming that the distribution of the share prices follows a normal distribution, test whether the mean share price is still the same at 1% level of significance?

Solution: Here, we wish to test that the mean share price (μ) of companies in the Pharma sector is still Rs.70 besides all changes. So our claim is $\mu = 70$ and its complement is $\mu \neq 70$. Since the claim contains the equality sign so we can take the claim as the null hypothesis and complement as the alternative hypothesis. Thus,

$$H_0 : \mu = \mu_0 = 70 \quad [\text{mean share price of companies is still Rs. 70}]$$

$$H_1 : \mu \neq \mu_0 = 70 \quad [\text{mean share price of companies is not still Rs. 70}]$$

Since the alternative hypothesis is two-tailed so the test is two-tailed.

Here, we want to test the hypothesis regarding population mean when population SD is unknown. Also, the sample size is small $n = 10$ ($n < 30$) and the population under study is normal, so we can go for the t-test for testing the hypothesis about the population mean.

For testing the null hypothesis, the test statistic t is given by

$$t = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim t_{(n-1)} \quad \dots (1)$$

Calculation for $\bar{X}$ and S :

S. No.	Sample value (X)	Deviation $d = (X-a), a = 70$	d^2
1	70	0	0
2	76	6	36
3	75	5	25
4	69	-1	1
5	70	0	0
6	72	2	4
7	68	-2	4
8	65	-5	25
9	75	5	25
10	72	2	4
Total		12	124

The assumed value of a is 70.

From the above calculation, we have

$$\bar{X} = a + \frac{1}{n} \sum d = 70 + \frac{1}{10} \times 12 = 71.2$$

$$S^2 = \frac{1}{n-1} \left[\sum d^2 - \frac{(\sum d)^2}{n} \right]$$

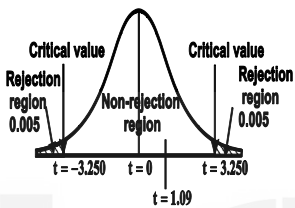
$$= \frac{1}{10-1} \left[124 - \frac{(12)^2}{10} \right] = \frac{1}{9} \left[124 - \frac{144}{10} \right] = 12.18$$

$$\Rightarrow S = \sqrt{12.18} = 3.49$$

Putting the values in equation (1), we have

$$t = \frac{71.2 - 70}{3.49 / \sqrt{10}} = \frac{1.2}{1.10} = 1.09$$

Fig. 10.11



The critical (tabulated) values of the test statistic for the two-tailed test corresponding $(n-1) = 9$ df at 1% level of significance are

$$\pm t_{(n-1), \alpha/2} = \pm t_{(9), 0.005} = \pm 3.250.$$

Since the calculated value of the test statistic $t (= 1.09)$ is less than the critical value $(= 3.250)$ and greater than the critical value $(= -3.250)$, that means the calculated value of t lies in the non-rejection region as shown in Fig. 10.11. So we do not reject the null hypothesis i.e. we support the claim at 1% level of significance.

Decision according to p-value:

Since the calculated value of the test statistic t is based on 9 df, therefore, we use row for 9 df in the t -table (Table V in Appendix) and move across this row to find the values in which the calculated t -value falls. Since all values in this row are greater than the calculated t -value 1.09 and the smallest value is 1.383 corresponding to one-tailed area $\alpha = 0.10$, therefore, the p -value is greater than 0.10, that is,

$$p\text{-value} > 0.10$$

Since the test is two-tailed so

$$p\text{-value} > 2 \times 0.10 = 0.20$$

Since the p -value $(= 0.20)$ is greater than $\alpha (= 0.01)$ so we do not reject the null hypothesis at 1% level of significance.

Thus, we conclude that the sample fails to provide us sufficient evidence against the claim so may assume that the mean share price is still Rs. 70.

Check Your Progress 3

- 1) A tyre manufacturer claims that the average life of a particular category of his tyre is 18000 km when used under normal driving conditions. A random sample of 16 tyres was tested. The mean and SD of life of the tyres in the sample were 20000 km and 6000 km respectively. Assuming that the life of the tyres is normally distributed, test the claim of the manufacturer at 1% level of significance using the appropriate test.

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- 2) It is known that the average weight of cadets of a centre follows a normal distribution. Weights of 10 randomly selected cadets from the same centre are as given below:

48, 50, 62, 75, 80, 60, 70, 56, 52, 77

Can we say that the average weight of all cadets of the centre from which the above sample was taken is equal to 60 kg at 5% level of significance?

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With this, we end this unit. Let us summarise what we have discussed in this unit.

10.9 SUMMARY

In this unit, we have covered the following points:

1. Statistical hypothesis, null hypothesis, alternative hypothesis.
2. Critical region and Type-I and Type-II errors.
3. One-tailed and two-tailed tests.
4. General procedure of testing a hypothesis.
5. Level of significance and Concept of p-value.
6. Applying the Z-test for testing the hypothesis about the population mean and the difference of two population means.
7. Applying the Z-test for testing the hypothesis about the population proportion.
8. Testing of hypothesis for population mean using t-test.

10.10 SOLUTIONS / ANSWERS

Check Your Progress 1

- 1) Here, we wish to test the claim of the company that the average life of its car tyres is 50000 miles so

Claim: $\mu = 50000$ miles and complement: $\mu \neq 50000$ miles

Since claim contains equality sign so we take claim as the null hypothesis and complement as the alternative hypothesis i.e.

$H_0: \mu = 50000$ miles

$H_1: \mu \neq 50000$ miles

- 2) (iii) Here, the doctor wants to test whether new medicine is more effective for controlling blood pressure than old medicine so

Claim: $\mu_1 > \mu_2$ and complement: $\mu_1 \leq \mu_2$

Since complement contains equality sign so we take complement as the null hypothesis and claim as the alternative hypothesis i.e.

$H_0: \mu_1 \leq \mu_2$ and $H_1: \mu_1 > \mu_2$

(iv) Here, the economist wants to test whether the variability in incomes differ in two populations so

Claim: $\sigma_1^2 \neq \sigma_2^2$ and complement: $\sigma_1^2 = \sigma_2^2$

Since complement contains equality sign so we take complement as the null hypothesis and claim as the alternative hypothesis i.e.

$$H_0: \sigma_1^2 = \sigma_2^2 \quad \text{and} \quad H_1: \sigma_1^2 \neq \sigma_2^2$$

(v) Here, psychologist wants to test whether the proportion of literates between two groups of people is same so

Claim: $P_1 = P_2$ and complement: $P_1 \neq P_2$

Since claim contains equality sign so we take claim as the null hypothesis and complement as the alternative hypothesis i.e.

$$H_0: P_1 = P_2 \quad \text{and} \quad H_1: P_1 \neq P_2$$

- 3) Since alternative hypothesis $H_1: \theta \neq 60$ is two-tailed so critical region lies in two-tails.
- 4) Since level of significance is the probability of type-I error so in this case level of significance is 0.05 or 5%.
- 5) Here, the alternative hypothesis is two-tailed therefore, the test will be two-tailed.
- 6) Whether the test of testing a hypothesis is one-tailed or two-tailed depends on the alternative hypothesis. So the correct option is (ii).

Check Your Progress 2

- 1) We are given that

$$n = 900, \bar{X} = 3.4 \text{ cm}, \mu_0 = 3.25 \text{ cm and } \sigma = 2.61 \text{ cm}$$

Here, we wish to test that the sample comes from a large population of bolts with mean (μ) 3.25cm. So our claim is $\mu = 3.25\text{cm}$ and its complement is $\mu \neq 3.25\text{cm}$. Since the claim contains the equality sign so we can take the claim as the null hypothesis and the complement as the alternative hypothesis. Thus,

$$H_0: \mu = \mu_0 = 3.25 \text{ and } H_1: \mu \neq 3.25$$

Since the alternative hypothesis is two-tailed, so the test is two-tailed.

Here, we want to test the hypothesis regarding population mean when population SD is unknown, so we should use the t-test if the population of bolts known to be normal. But it is not the case. Since the sample size is large ($n > 30$) so we can go for the Z-test instead of the t-test as an approximate. So the test statistic is given by

$$\begin{aligned} Z &= \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}} \\ &= \frac{3.40 - 3.25}{2.61 / \sqrt{900}} = \frac{0.15}{0.087} = 1.72 \end{aligned}$$

The critical (tabulated) values for the two-tailed test at 5% level of significance are $\pm z_{\alpha/2} = \pm z_{0.025} = \pm 1.96$.

Since the calculated value of the test statistic $Z (= 1.72)$ is less than the critical value $(= 1.96)$ and greater than the critical value $(= -1.96)$, that means it lies in the non-rejection region, so we do not reject the null hypothesis i.e. we support the claim at 5% level of significance.

Thus, we conclude that the sample does not provide us sufficient evidence against the claim so we may assume that the sample comes from the population of bolts with mean 3.25cm.

2) Given that

$$n_1 = 200, \quad \bar{X} = 1300, \quad S_1 = 41;$$

$$n_2 = 200, \quad \bar{Y} = 1280, \quad S_2 = 46$$

Here, we want to test that there is a significant difference in the mean duration of their lives of two brands of electric bulbs. If μ_1 and μ_2 denote the mean lives of two brands of electric bulbs respectively then our claim is $\mu_1 \neq \mu_2$ and its complement is $\mu_1 = \mu_2$. Since the complement contains the equality sign so we can take the complement as the null hypothesis and the claim as the alternative hypothesis. Thus,

$$H_0 : \mu_1 = \mu_2 \text{ and } H_1 : \mu_1 \neq \mu_2$$

Since the alternative hypothesis is two-tailed so the test is two-tailed.

We want to test the null hypothesis regarding the equality of two population means. The standard deviations of both populations are unknown so we should go for the t-test if the population of difference is known to be normal. But it is not the case. Since sample sizes are large (n_1 , and $n_2 > 30$) so we go for the Z-test.

So for testing the null hypothesis, the test statistic Z is given by

$$\begin{aligned} Z &= \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} \\ &= \frac{1300 - 1280}{\sqrt{\frac{(41)^2}{200} + \frac{(46)^2}{200}}} = \frac{20}{\sqrt{8.41 + 10.58}} = \frac{20}{4.36} = 4.59 \end{aligned}$$

The critical (tabulated) values for the two-tailed test at 1% level of significance are $\pm z_{\alpha/2} = \pm z_{0.005} = \pm 2.58$.

Since the calculated value of the test statistic $Z (= 4.59)$ is greater than the critical value $(= \pm 2.58)$, that means it lies in the rejection region, so we reject the null hypothesis and support the alternative hypothesis i.e. support the claim at 1% level of significance.

Thus, we conclude that the samples do not provide us sufficient evidence against the claim so there is a significant difference in the mean duration of their lives of two brands of electric bulbs.

3) We define our success and failure as a patient is survived and not survived. Here, we are given that

$$n = 200$$

X = Number of survived patients who are given a particular injection

$$= 180$$

p = Sample proportion of survived patients who are given a particular injection

$$= \frac{X}{n} = \frac{180}{200} = 0.9$$

$$P_0 = 80\% = \frac{80}{100} = 0.80 \Rightarrow Q_0 = 1 - P_0 = \frac{80}{100} = 0.20$$

Here, we want to test that the survival rate of the patients is more than 80%. If P denotes the proportion of survival patients then our claim is $P > 0.80$ and its complement is $P \leq 0.80$. Since complement contains the equality sign so we can take the complement as the null hypothesis and the claim as the alternative hypothesis. Thus,

$$H_0 : P \leq P_0 = 0.80 \text{ and } H_1 : P > 0.80$$

Since the alternative hypothesis is right-tailed so the test is right-tailed.

Before proceeding further, first, we have to check whether the conditions of normality meet or not.

$$\therefore np = 200 \times 0.9 = 180 > 5$$

$$nq = 200 \times (1 - 0.9) = 200 \times 0.1 = 20 > 5$$

We see that the conditions of normality meet, so we can go for the Z -test.

For testing the null hypothesis, the test statistic Z is given by

$$\begin{aligned} Z &= \frac{p - P_0}{\sqrt{\frac{P_0 Q_0}{n}}} \\ &= \frac{0.9 - 0.8}{\sqrt{\frac{(0.8)(0.2)}{200}}} = \frac{0.1}{0.0283} = 3.53 \end{aligned}$$

The critical (tabulated) value for the right-tailed test at 5% level of significance is $z_\alpha = z_{0.05} = 1.645$.

Since the calculated value of the test statistic Z ($= 3.53$) is greater than the critical value ($= 1.645$), that means it lies in the rejection region, so we reject the null hypothesis and support the alternative hypothesis i.e. support our claim at 5% level of significance.

Thus, we conclude that the sample fails to provide us sufficient evidence against the claim so we may assume that the survival rate is greater than 80% in the population using that injection.

Check Your Progress 3

1) Here, we are given that

$$n = 16, \mu_0 = 18000, \bar{X} = 20000, S = 6000$$

Here, we want to test that the manufacturer's claim is true that the average life (μ) of tyres is 18000 km. So claim is $\mu = 18000$ and its complement is $\mu \neq 18000$. Since the claim contains the equality sign so

we can take the claim as the null hypothesis and complement as the alternative hypothesis. Thus,

$$H_0: \mu = \mu_0 = 18000 \quad [\text{average life of tyres is } 18000 \text{ km}]$$

$$H_1: \mu \neq 18000 \quad [\text{average life of tyres is not } 18000 \text{ km}]$$

Here, population SD is unknown and the population under study is given to be normal. So we can go for the t-test.

For testing the null hypothesis, the test statistic t is given by

$$t = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} = \frac{20000 - 18000}{6000/\sqrt{16}} = \frac{2000}{1500} = 1.33$$

The critical value of the test statistic t for the two-tailed test corresponding $(n-1) = 15$ df at 1% level of significance are

$$\pm t_{(15), 0.005} = \pm 2.947.$$

Since the calculated value of the test statistic t (= 1.33) is less than the critical (tabulated) value (= 2.947) and greater than critical value (= -2.947), that means the calculated value of the test statistic lies in the non-rejection region, so we do not reject the null hypothesis i.e. we support the manufacturer's claim at 1% level of significance.

Thus, we conclude that the sample fails to provide sufficient evidence against the claim so we may assume that manufacturer's claim is true.

- 2) Here, we want to test that the average weight (μ) of all cadets of the centre is 60 kg. So our claim is $\mu = 60$ and its complement is $\mu \neq 60$. Since the claim contains the equality sign so we can take the claim as the null hypothesis and complement as the alternative hypothesis. Thus,

$$H_0 : \mu = \mu_0 = 60 [\text{average weight of all cadets is } 60 \text{ kg}]$$

$$H_1 : \mu \neq 60 \quad [\text{average weight of all cadets is not } 60 \text{ kg}]$$

Since the alternative hypothesis is two-tailed so the test is two-tailed.

Here, population SD is unknown and the population under study is given to be normal. So we can go for the t-test.

For testing the null hypothesis, the test statistic t is given by

$$t = \frac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim t_{(n-1)} \quad \text{under } H_0 \quad \dots (5)$$

Before moving further, first, we have to calculate the value of $\bar{X}$ and S.

Calculation for $\bar{X}$ and S:

Sample value (X)	$(x - \bar{X})$	$(x - \bar{X})^2$
48	-15	225
50	-13	169
62	-1	1
75	12	144
80	17	289
60	-3	9
70	7	49
56	-7	49
52	-11	121

77	14	196
$\sum X = 630$		$\sum (X - \bar{X})^2 = 1252$

From the above calculation, we have

$$\bar{X} = \frac{1}{n} \sum X = \frac{1}{10} \times 630 = 63$$

$$S^2 = \frac{1}{n-1} \sum (X - \bar{X})^2 = \frac{1}{10-1} \times 1252 = 139.11$$

$$\Rightarrow S = \sqrt{139.11} = 11.79$$

Putting the values in equation (5), we have

$$t = \frac{63 - 60}{11.79 / \sqrt{10}} = \frac{3}{3.73} = 0.80$$

The critical values of the test statistic t for the two-tailed test corresponding $(n-1) = 9$ df at 5% level of significance are $\pm t_{(9), 0.025} = \pm 2.262$.

Since the calculated value of the test statistic t ($= 0.80$) is less than the critical value ($= 2.262$) and greater than the critical value ($= -2.262$), that means the calculated value of the test statistic t lies in the non-rejection region so we do not reject H_0 i.e. we support the claim at 5% level of significance.

Thus, we conclude that the sample fails to provide sufficient evidence against the claim so we may assume that the average weight of all the cadets of the given centre is 60 kg.

10.11 REFERENCES

IGNOU Course Material

1. Post-Graduate Diploma in Applied Statistics (PGDAST), MST 004: Statistical Inference, Block 3: Testing of Hypothesis UNIT 9: Concepts of Testing of Hypothesis (for Sections 10.2 to 10.4 and related portions in Sections 10.0 , 10.1, 10.9, 10.10 and Check Your Progress questions)
2. Post-Graduate Diploma in Applied Statistics (PGDAST), MST 004: Statistical Inference, Block 3: Testing of Hypothesis UNIT 10: Large Sample Tests (for Sections 10.5 to 10.7 and related portions in Sections 10.0 , 10.1, 10.9, 10.10 and Check Your Progress questions)
3. Post-Graduate Diploma in Applied Statistics (PGDAST), MST 004: Statistical Inference, Block 3: Testing of Hypothesis UNIT 11: Small Sample Tests (for Sections 10.8 and related portions in Sections 10.0 , 10.1, 10.9, 10.10 and Check Your Progress questions)

Reference Book

4. "Introduction to Probability and Statistics for Engineers and Scientists", 3rd Edition, Sheldon M. Ross, Elsevier Academic Press, 2004